Notes from FE Review for mathematics, University of Kentucky

Dr. Alan Demlow*

March 19, 2012

Introduction

These notes are based on reviews for the mathematics portion of the Fundamentals of Engineering exam given at the University of Kentucky. As you study these notes, keep in mind that the FE Supplied Reference Handbook contains many of the basic formulas covered here. You should also familiarize yourself with the reference handbook. While studying, you should concentrate on ensuring that you understand the ideas behind the formulas and can work relevant problems. Also, these notes are based on a two-hour review session, and it is not possible to cover all of the concept from the FE exam during that time period. Please make sure you fill in any gaps! Finally, there are of course other resources available to help you study. One example is the FE Exam review site at the University of Oklahoma: `http://www.feexam.ou.edu`.

1 Algebra

1.1 Quadratic equations

A quadratic equation is given by $ax^2 + b^x + c$. The quadratic formula for finding roots of this equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

*Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506–0027 (alan.demlow@uky.edu)

The sign of the discriminant $b^2 - 4ac$ determines what types of roots this equation will have. If $b^2 - 4ac > 0$, there will be two distinct real roots. If $b^2 - 4ac = 0$, there will be one repeated real root. If $b^2 - 4ac < 0$, there will be two complex roots which are conjugates of each other. For example, if we wish to solve $3x^2 + 2x + 1 = 0$, then the discriminant is $2^2 - 4 \cdot 3 \cdot 1 = 4 - 12 = -8$, so we expect to have two complex roots. Indeed,

$$x = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \frac{2\sqrt{-2}}{2} = -1 \pm i\sqrt{2}.$$

1.2 Logarithms

The logarithm function $y = \log_a x$ is the inverse of the exponential function a^x, so that $y = \log_a x$ if and only if $x = a^y$. (Here $a > 0$ is the *base* of the logarithm.) Special cases of the logarithm include base $a = 10$, in which case we write $\log_{10} x = \log x$; and $a = e$, in which case we write $\log_e x = \ln x$. ln is called the natural logarithm, and $e = 2.71828....$

Here are several important properties of logarithms:

Changing bases:	$\log_a x = \dfrac{\log_b x}{\log_b a}.$
Logs of products:	$\log_a(xy) = \log_a x + \log_a y.$
Logs of differences:	$\log_a \dfrac{x}{y} = \log_a x - \log_a y.$
Logs of exonentials:	$\log_a x^y = y \log_a x.$
Other rules:	$\log_a 1 = 0, \quad \log_a a = 1, \quad \log_a a^x = x \log_a a = x.$

For example, $\ln x = \frac{\log x}{\log e}$. Also, $\log 1000^3 = 3 \log 1000 = 3 \log 10^3 = 9 \log 10 = 9$. Finally, $\ln e^{x-y} = (x - y) \ln e = (x - y)$.

1.3 Trigonometry

Be sure you understand the basic definitions of sin and cos from right triangle trigonometry. The four other main trig functions can be derived from these: $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x} = 1/\tan x$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$. The FE Supplied Reference Handbook also contains a large number of trig identities, starting with the most important one: $\sin^2 x + \cos^2 x = 1$. It is of course not necessary to memorize these, but you probably will want to familiarize yourself with them so you know where to find them in case you need them.

1.4 Complex numbers

A complex number is given by $a+ib$, where a, b are real numbers and $i^2 = -1$ (or, $i = \sqrt{-1}$). Complex numbers may be multiplied by using normal rules of algebra ("first-outer-inner-last", for example) and then using the identity $i^2 = -1$. For example,

$$(2+3i)(1+2i) = 2{\cdot}1+2{\cdot}(2i)+3i{\cdot}1+(3i)(2i) = 2+7i+6i^2 = 2+7i-6 = -4+7i.$$

The *complex conjugate* of a complex number $a+ib$ is given by $\overline{a+ib} = a-ib$. In order to divide two complex numbers, we multipliy top and bottom by the complex conjugate of the denominator. For example,

$$\frac{2+3i}{1+2i} = \frac{2+3i}{1+2i}\frac{1-2i}{1-2i} = \frac{2-i-6i^2}{1-4i^2} = \frac{8-i}{5} = \frac{8}{5} - \frac{1}{5}i.$$

Note also that $(a+ib)(\overline{a+ib}) = a^2+b^2$.

Complex numbers may also be represented using polar coordinates. We write $z = a+ib = re^{i\theta}$, where $e^{i\theta}$ may be computed using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$; see Figure 1. The conversion between standard

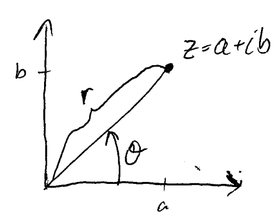

Figure 1: Polar representation of a complex number. {fig1-4}

and polar representations may be carried out exactly as for the conversion between the Cartesian coordinate (a, b) and its polar representation (r, θ). That is, $r = \sqrt{a^2+b^2}$, and θ is any angle terminating in the same quadrant as (a, b) for which $\tan\theta = \frac{b}{a}$. (Recall that we may *not* simply write $\theta = \arctan\frac{b}{a}$, since the range of arctan is $-\frac{\pi}{2} < y < \frac{\pi}{2}$, but (a, b) may also lie in the second or third quadrant.) For example, let $z = (2-2i)$. Then $r = \sqrt{2^2+(-2)^2} = \sqrt{8}$. Also, θ satisfies $\tan\theta = \frac{2}{-2} = -1$, and θ lies in the same quadrant as $(2, -2)$, which is the fourth quadrant. Thus $\theta = -\frac{\pi}{4}$; see Figure 2. We then write $2-2i = \sqrt{8}e^{-i\pi/4}$. We may carry out the following multiplication as follows:

$$(2-2i)\sqrt{32}e^{i\pi/4} = \sqrt{8}e^{-i\pi/4}\sqrt{32}e^{\pi/4} = \sqrt{256}e^{-i\pi/4+i\pi/4} = \sqrt{256}e^0 = 16.$$

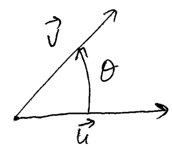

Figure 2: Polar representation of $z = 2 - 2i$. {fig1-4-2}

2 Vectors

2.1 The dot product

A vector $\vec{u}$ in three space dimensions is represented as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} = \langle u_1, u_2, u_3\rangle$. The dot product is defined by $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, where $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ is the length of $|\vec{u}|$ and θ is the angle between u_1 and u_2 (see Figure 3). Thus the dot product of two vectors is a *scalar*. Note

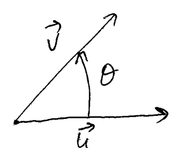

Figure 3: The angle between two vectors. {fig2-1}

also that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}.$$

We may alternatively write

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note also that the dot product is commutative, that is, $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$. Recall also that two nonzero vectors $\vec{u}, \vec{v}$ are *perpendicular* if and only if $\vec{u}\cdot\vec{v} = 0$.

2.2 The cross product

Geometrically, the cross product $\vec{u}\times\vec{v}$ of two three-vectors is a *vector* which is perpendicular to the plane in which $\vec{u}$ and $\vec{v}$ lie; $\vec{u}\times\vec{v}$ has orientation given by the right hand rule (curl the fingers of your right hand through $\vec{u}$, then through $\vec{v}$, and your thumb will point in the direction of $\vec{u}\times\vec{v}$). This also leads us to observe that the cross product is NOT commutative;

instead we have $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. The cross product may be computed using determinants:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

The area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ is given by $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$, where θ is again the angle between $\vec{u}$ and $\vec{v}$ (see Figure 4). This

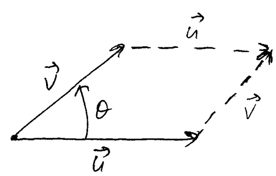

Figure 4: Parallelogram formed by two vectors. {fig2-2}

leads to the observation that two nonzero vectors $\vec{u}$ and $\vec{v}$ are *parallel* if and only if their cross product is the zero vector. In addition, the volume of the parallelepiped spanned by three vectors $\vec{u}, \vec{v}, \vec{w}$ is given by $|\vec{w} \cdot (\vec{u} \times \vec{v})|$. Example: Find the volume of the parallelepiped spanned by

$$\vec{u} = \vec{i} + \vec{j} - \vec{k}, \quad \vec{v} = 2\vec{i} + 3\vec{j} = 4\vec{k}, \quad \vec{w} = 4\vec{i} + \vec{j} - \vec{k}.$$

To solve, we first compute $\vec{u} \times \vec{v}$:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}.$$

$$= \vec{i}(1 \cdot 4 - 3 \cdot 1) - \vec{j}(1 \cdot 4 - 2 \cdot 1) + \vec{k}(1 \cdot 3 - 2 \cdot 1) = \vec{i} - 2\vec{j} + \vec{k}.$$

The volume of the parallelepiped is then

$$|\vec{w} \cdot (\vec{u} \times \vec{v})| = |(4\vec{i} + \vec{j} - \vec{k}) \cdot (\vec{i} - 2\vec{j} + \vec{k}) = 4 \cdot 1 + 1 \cdot (-2) - 1 \cdot 1 = 4 - 2 - 1 = 1.$$

3 Matrices

3.1 Matrix basics and matrix multiplication

An $m \times n$ matrix is an array of numbers having m rows and n columns. In order to find the product AB of two matrices A and B, B must have the same number of rows as A has columns. That is, A must be $m \times n$ and B $n \times p$, and the resulting matrix AB is $m \times p$. The ij-th entry of AB is obtained by taking the dot product of the i-th row of A with the j-th column of B. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} \langle 1,2,3\rangle \cdot \langle 1,3,5\rangle & \langle 1,2,3\rangle \cdot \langle 2,4,6\rangle \\ \langle 4,5,6\rangle \cdot \langle 1,3,5\rangle & \langle 4,5,6\rangle \cdot \langle 2,4,6\rangle \end{bmatrix}$$
$$= \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}.$$

Note also that the product of a 2×3 matrix with a 3×2 matrix is a 2×2 matrix. Note also that in general $AB \neq BA$. In fact, it may be possible to compute AB but not BA.

The identity matrix I is a square matrix with the property that $IA = A$ and $AI = A$ whenever these expressions make sense (i.e., when the dimensions of A and I match appropriately). The 3×3 identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transpose of a matrix $A = [a_{ij}]$ is given by $A^T = [a_{ji}]$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

The determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The determinant of a 3×3 matrix (given by expansion along the first row) is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}.$$

The *inverse* A^{-1} of a square matrix A satisfies $AA^{-1} = A^{-1}A = I$. Note that A may not have an inverse even if A is nonzero; in fact A^{-1} exists if and only if $\det A \neq 0$. Cofactors may be used to compute the inverses of matrices. Given a 3×3 matrix $A = [a_{ij}]$, the cofactor of entry a_{ij} is $c_{ij} = (-1)^{i+j} \det A_{ij}$, where A_{ij} is the 2×2 matrix obtained by deleting the i-th row and j-th column of A. Then $A^{-1} = \frac{1}{\det A}[c_{ij}]^T$.

3.2 Solution of linear systems of equations

Matrices may be used to solve linear systems via Gaussian elimination. For example, in order to solve

$$\begin{aligned} 2x + 4y + 6z &= 0, \\ 4x + 5y + 6z &= 3, \\ 4x + 8y + 9z &= 6, \end{aligned}$$

we may place the coefficients of the system into a matrix and then perform elementary row operations (multiplying rows by constants and replacing rows with linear combinations) as follows:

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 4 & 5 & 6 & 3 \\ 4 & 8 & 9 & 6 \end{array}\right] \xrightarrow[R3-2R1\to R3]{\substack{R1/2\to R1 \\ R2-2R1\to R2}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & -3 & 6 \end{array}\right] \xrightarrow[-R3/3\to R3]{-R2/3\to R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[R2-2R3\to R2]{R1-2R2\to R1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R1+R3\to R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Thus the solution to the linear system is $x = 0$, $y = 3$, $z = -2$.

4 Differential calculus

4.1 Derivatives and slope

The derivative of a function $y = f(x)$ at x is defined by

$$y'(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

where $\Delta y = f(x + \Delta x) - f(x)$. $f'(x)$ is the slope of the tangent line to the graph $y = f(x)$ at x. For example, let $f(x) = \tan x + \ln x$. Then $f'(x) = \sec^2 x + \frac{1}{x}$. The slope of the tangent line at $x = \pi$ is $\sec^2 \pi + \frac{1}{\pi} = 1 + \frac{1}{\pi}$. We may find the equation of the tangent line by using the point-slope formula $y - y_0 = m(x - x_0)$. $y_0 = f(\pi) = \tan \pi + \ln \pi = \ln \pi$. Thus $y = \ln \pi + (1 + \frac{1}{\pi})(x - \pi)$ is the equation of the tangent line.

4.2 Maxima and minima of functions; inflection points

Let $y = f(x)$. We say that $f(x)$ is increasing on an interval (a, b) if $f(y) > f(x)$ whenever $y > x$. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b). Similarly, f is decreasing on (a, b) if $f'(x) < 0$ on (a, b).

A function $f(x)$ has a local maximum at a if $f(x) \leq f(a)$ for all x near a; f has a local minimum at a if $f(x) \geq f(a)$ for all x near a. The second derivative test states that $f(x)$ has a local maximum (minimum) at a if $f'(a) = 0$ and $f''(a) < 0$ ($f''(a) > 0$)).

A function $f(x)$ has an *absolute* maximum (minimum) at $c \in [a, b]$ if $f(c) \geq f(x)$ ($f(c) \leq f(x)$) for all $x \in [a, b]$. A continuous function f always takes on an absolute minimum and maximum on a *closed* interval $[a, b]$. The closed interval tests states that the absolute minimum and maximum of f may be found by comparing the values of f at the critical points (places where $f'(x) = 0$ or $f'(x)$ doesn't exist) and endpoints a, b of $[a, b]$.

For example, let $f(x) = -x^3 + 3x$. We have $f'(x) = -3x^2 + 3$. Setting $f'(x) = -3(x^2 - 1) = 0$, we see that f has critical points at $x = \pm 1$. Also, $f''(x) = -6x$, so $f''(-1) = 6 > 0$. Thus by the second derivative test, f has a *local minimum* at the critical point $x = -1$. Similarly, $f''(1) = -6 < 0$, so f has a *local maximum* at $x = 1$. If we wish to find the *absolute extreme values* of f on the closed interval $[-2, 3]$, we use the closed interval test by comparing the values of f at the critical points ± 1 and endpoints $-2, 3$. $f(-1) = 1 - 3 = -2$ and $f(1) = -1 + 3 = 2$. Also, $f(-2) = 8 - 6 = 2$, and $f(3) = -27 + 9 = -18$. Thus the absolute maximum of f on $[-2, 3]$ is 2 (taken on at 1 and -2), and the absolute minimum is -18 (taken on at 3). See Figure 5.

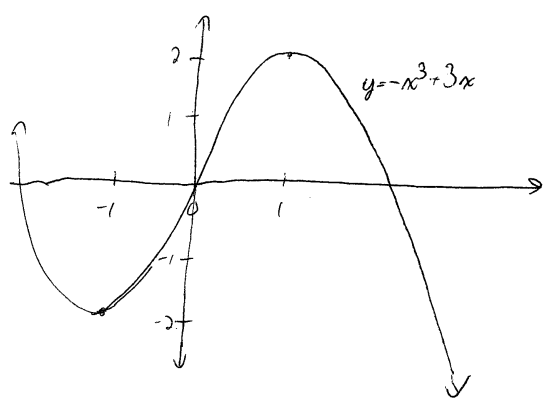

Figure 5: Finding minimum and maximum values. {fig5}

f has an inflection point at c if f'' changes sign at c. We check for inflection points by looking for points c where $f''(c) = 0$ or $f''(c)$ doesn't exist. In the above example $f(x) = -x^3 + 3x$, we have $f''(x) = -6x$, so the point $c = 0$ is a candidate for an inflection point. When $x < 0$, $f''(x) > 0$, and $f''(x) < 0$) when $x > 0$. Thus f'' changes sign (and thus also concavity) at $c = 0$, and $c = 0$ is an inflection point of f.

4.3 Partial derivatives

In order to take partial derivative of a function of several variables with respect to one of the variables x, we treat all other variables as constants and differentiate with respect to x. For example, let $P = P(R,S,T) = R^{1/4}T\sin(2S)$. Then $\frac{\partial P}{\partial R} = \frac{1}{4}R^{-3/4}T\sin(2S)$, $\frac{\partial P}{\partial S} = 2R^{1/4}T\cos(2S)$, and $\frac{\partial P}{\partial T} = R^{1/4}\sin(2S)$.

5 Integral calculus

5.1 Indefinite integrals

An indefinite integral (or antiderivative) of a given function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. $F(x)$ is specified only up to an arbitrary constant. For example,

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \; n \neq -1, \qquad \int \frac{\mathrm{d}x}{x} = \ln|x| + C.$$

A number of other indefinite integrals may be found in the FE Reference book.

5.2 Definite integrals and integration techniques

A definite integral is defined as a limit of Riemann sums: $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x$, where $x_i = a + i\Delta x$, $x_i^* \in [x_{i-1}, x_i]$, and $\Delta x = \frac{b-a}{n}$. See Figure 6. Note that $\int_a^b f(x)\,\mathrm{d}x$ may be interpreted as the area under the curve $y = f(x)$ when f is positive, and as the *net* area in the general case.

As an example, we compute the definite integral $\int_0^2 xe^{x^2}\,\mathrm{d}x$ using substitution. Let $u = x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x$. Note that we must also change the limits of integration: $u(0) = 0^2 = 0$ and $u(2) = 2^2 = 4$. Thus

$$\int_0^2 xe^{x^2}\,\mathrm{d}x = \int_0^4 \frac{1}{2}e^u\,\mathrm{d}u = \frac{1}{2}e^u|_0^4 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1).$$

Another important integration technique is integration by parts (we only do an example of an indefinite integral). The formula is $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$. To find $\int xe^x\,\mathrm{d}x$, we let $u = x$ (since differentiating x results in a simpler

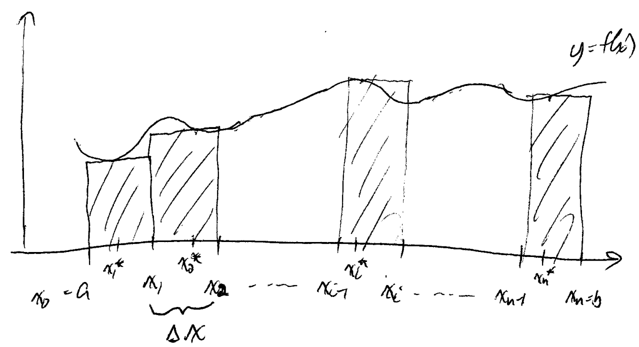

Figure 6: Definition of a Riemann sum. {fig6}

expression) and $v = e^x\,\mathrm{d}x$ (since we may antidifferentiate e^x easily). Then $\mathrm{d}u = \mathrm{d}x$, and $v = e^x$. So,

$$\int xe^x\,\mathrm{d}x = uv - \int v\,\mathrm{d}u = xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x.$$

Partial fractions are also sometimes used in order to compute definite and indefinite integrals. For example, suppose we wish to integrate $\frac{2x+1}{(x+2)(x+1)}$. In this (simple) case, partial fractions can be thought of as "undoing" the standard cross-multiply-and-divide operation:

$$\frac{2x+1}{(x+2)(x+1)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

Cross multiplying yields

$$2x + 1 = A(x+2) + B(x+1) = x(A+B) + (2A+B).$$

Thus we must have $A + B = 2$, $2A + B = 1$. Solving for A and B yields $A = -1$, $B = 3$. Then

$$\int \frac{2x+1}{(x+2)(x+1)}\,\mathrm{d}x = \int \left(\frac{-1}{x+1} + \frac{3}{x+2} \right) \mathrm{d}x = -\ln|x+1| + 3\ln|x+2| + C.$$

If the denominator has nonsimple roots or irreducible quadratic expressions, then there other rules for partial fractions expansions. For example, we write:

$$\frac{2x+1}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)}, \qquad \frac{3}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}.$$

5.3 Areas and averages

As an example of an area problem, we find the area between the curves $y = x$ and $y = \sqrt{x}$. Note that these curves intersect at $x = 0$ and $x = 1$, and that $\sqrt{x} > x$ for $0 < x < 1$. Thus the area between the curves is $\int_0^1 (\sqrt{x} - x)\,\mathrm{d}x = (\frac{2}{3}x^{3/2} - \frac{1}{2}x^2)|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$. See Figure 7.

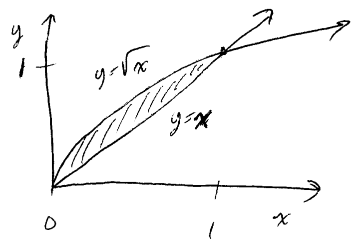

Figure 7: The area between two curves. {fig7}

As a final application of definite integration, we note that the average value of a function $f(x)$ between $x = a$ and $x = b$ is $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$.

6 Differential equations

6.1 First-order ordinary differential equations

There are several types of first-order ordinary differential equations that can be solved by hand; we consider two. *Linear* equations have the form $y' + p(x)y = f(x)$ and may be solved using the method of integrating factors. The integrating factor is define by $\mu(x) = e^{\int p(x)\,\mathrm{d}x}$. Then using the Fundamental Theorem of Calculus, we find that:

$$(\mu(x)y)' = \mu(x)f(x) \quad \Rightarrow \quad \mu y = \int (\mu f)\,\mathrm{d}x, \text{ or } y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,\mathrm{d}x.$$

For example, consider the initial value problem $y' + 2xy = x$, $y(0) = 1$. Here $p(x) = 2x$, so $\mu(x) = e^{\int 2x\,\mathrm{d}x} = e^{x^2}$. Thus $(e^{x^2}y)' = xe^{x^2}$, and

$$e^{x^2}y = \int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}e^{x^2} + C.$$

Solving, we have $y = \frac{1}{2} + Ce^{-x^2}$. Also, $1 = y(0) = \frac{1}{2} + Ce^0$, so $C = 1 - \frac{1}{2} = \frac{1}{2}$. Thus $y(x) = \frac{1}{2} + \frac{1}{2}e^{-x^2}$ solves the given initial value problem.

Separable equations have the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p(x)}{q(y)}$. Separating variables and integrating yields $\int q(y)\,\mathrm{d}y = \int p(x)\,\mathrm{d}x$. For example, consider the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = -2e^{2x}y^2$, $y(0) = 2$. Separating variables yields

$$\int \frac{\mathrm{d}y}{y^2} = \int -2e^{2x}\,\mathrm{d}x \quad \Rightarrow \quad -\frac{1}{y} = -e^{2x} + C.$$

Also, $y(0) = 2$, so $-\frac{1}{2} = -e^0 + C$, or $C = \frac{1}{2}$. Thus $-\frac{1}{y} = -e^{2x} + \frac{1}{2}$, or

$$y = \frac{1}{e^{2x} - \frac{1}{2}}.$$

6.2 Second-order ordinary differential equations

We first consider the second-order constant coefficient linear homogeneous equation $ay'' + by' + cy = 0$, where $a, b, c \in \mathbb{R}$. $y = e^{rx}$ is a solution to this equation if r is a root of the characteristic equation $ar^2 + br + c = 0$. There are three cases to consider:

1. The characteristic equation has two real roots $r_1 \neq r_2$, in which case the general solution is $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$ and the motion is said to be *overdamped*.

2. The characteristic equation has a single repeated real root r, in which case the general solution is $y(x) = c_1e^{rx} + c_2xe^{rx}$ and the motion is said to be *critically damped*.

3. The characteristic equation has two complex roots $r_\pm = \alpha \pm i\beta$, in which case the general solution is $y(x) = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$ and the motion is said to be *underdamped*.

As an example, consider the initial value problem $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 2$. The characteristic equation $r^2 + 2r + 1 = 0$ has a single repeated real root -1 (note the factorization $r^2 + 2r + 1 = (r + 1)^2$), so the general solution is $y(x) = c_1e^{-x} + c_2xe^{-x}$. Also, $1 = y(0) = c_1$. Then $y'(x) = -e^{-x} + c_2e^{-x} - c_2xe^{-x}$, so $2 = y'(0) = -1 + c_2$, and $c_2 = 3$. The solution to the given initial value problem is thus $y(x) = e^{-x} + 3xe^{-x}$.

Consider now the nonhomogeneous equation $ay'' + by' + cy = f(x)$. The general solution has the form $y = c_1y_1(x) + c_2y_2(x) + Y_p(x)$, where $Y_p(x)$ is any *particular* solution to the nonhomogeneous equation $ay'' + by' + cy = f(x)$ and $c_1y_1(x) + c_2y_2(x)$ is the *general* solution to the corresponding homogeneous equation $ay'' + by' + cy = 0$. For example, consider the equation

$y'' + y = e^{2x}$. Using the method of undetermined coefficients, we guess that a particular solution will have the form $Y_p(x) = Ae^{2x}$. Then $Y_p''(x) = 4Ae^{2x}$, so we must have $Y_p'' + Y_p = 4Ae^{2x} + Ae^{2x} = e^{2x}$. Thus $5A = 1$, or $A = \frac{1}{5}$, and our particular solution is $Y_p(x) = \frac{1}{5}e^{2x}$. The characteristic equation $r^2 + 1 = 0$ has roots $\pm i$, so the homogeneous equation $y'' + y = 0$ has general solution $c_1 \cos x + c_2 \sin x$. Finally, the general solution to $y'' + y = e^{2x}$ is $y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{5}e^{2x}$.

6.3 Laplace transforms

The Laplace transform is defined by $\mathcal{L}[f(t)] = \mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\, dt$. We also have $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$ and $\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$. A couple of standard Laplace transforms are $\mathcal{L}[1] = \frac{1}{s}$, and $\mathcal{L}[e^{ct}] = \frac{1}{s+c}$; others may be found in the FE-supplied reference handbook. As an example, we solve $y'' + 3y' + 2y = 1$, $y(0) = 0$, $y'(0) = 2$ using Laplace transforms. We have

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[1],$$

so

$$(s^2\mathcal{L}[y] - sy(0) - y'(0)) + 3(s\mathcal{L}[y] - y(0)) + 2\mathcal{L}[y] = \frac{1}{s}.$$

Inserting the initial conditions and collecting terms, we have

$$(s^2 + 3x + 2)\mathcal{L}[y] = \frac{1}{s} + 2 \quad \Rightarrow \quad \mathcal{L}[y] = \frac{2s+1}{s(s^2 + 3s + 2} = \frac{2s+1}{s(s+2)(s+1)}.$$

Using partial fractions, we write

$$\frac{2s+1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1},$$

so $A(s^2 + 3s + 2) + B(s^2 + s) + CC(s^2 + 2s) = 2s + 1$. Thus

$$\begin{aligned} A + B + C &= 0, \\ 3A + B + 2C &= 2, \\ A &= \frac{1}{2}. \end{aligned}$$

The solution to this system is $A = \frac{1}{2}$, $B = -\frac{3}{2}$, $C = 1$. Thus $\mathcal{L}[y] = \frac{1/2}{s} - \frac{3/2}{s+2} + \frac{1}{s+1}$, and

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \frac{1}{2} - \frac{3}{2}e^{-2t} + e^{-t}.$$